Combinatorics in Blackjack

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***Abstract*—Blackjack is a game that uses 1 deck of playing cards which consist of 52 cards. This game has simple rules so it's famous to be played in most casinos. This paper will focus on the making strategy based on the combinatorial probability of the card instead of the betting part of the game. Decision making in Blackjack has an integral part of the game, by counting the probability of the cards that the player gets can help even win a game of Blackjack. 1 Deck of playing cards always have a fixed amount of cards and a fixed value of each card. This paper will show how to play to the point of winning a game of Blackjack by simply counting the combinatorial probability of the cards.**

***Keywords*—Blackjack, card, combinatorics, possibilities.**

I. Introduction

The game Blackjack is a card game that can be traced from the 17th century in Spain [1]. Its rules were first written in the 19th century, when at the time it was famous with the name *Vingt-Un* ("Twenty-One"). Then the name changed to the familiar name Blackjack in the year 1899 [2]. The game has multiple variations of rules as the time progresses, but the main goal of this game is you have to accumulate value from the cards in your hand close to 21, but not over 21. Then you will compare your cards against the dealer cards, the one who gets the closest to the goal wins the round. This will be repeated multiple times until the player stops playing or the player is bankrupt if it was played in a casino. Blackjack is one of the most frequent games you can find in most casinos, because it involves luck and can become more fun when you put more stake in the game. Although the game is famous for its luck factor, you can still manage your odds while playing this game by utilizing mathematics, more specifically combinatorics.

Many authors try to explain the definition of combinatorics, so it's true definition isn't universally agreed upon [3]. In general, combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to many other areas of mathematics and has many applications ranging from logic to statistical physics, from evolutionary biology to computer science and for our cases, probability in 1 deck of cards.

1 Deck of playing cards consist of 52 cards, excluding the Joker cards. Regular playing cards always have cards valued 2-10, an Ace, and 3 face cards. All values come in 4 different types, such as spade, heart, club, and diamond.

There are some frequent basic terms that Blackjack uses. The first term is "Hit", this means you choose to take another card from the deck. The term "Busts" is used when the value of player hands exceeds 21, this will be explained more thoroughly in the Rules section. Then if you don't want to receive more cards from the deck you can use the term "Stand". After a player chooses to Stand, it ends the player's turn then moves on to the dealer's turn. The more complex terms will be explained in the following sections.

II. Basic Theory

Combinatorial is a field in mathematics to count an amount of all possible arrangements without counting every single one possibility [4]. In this field there are 2 basic rules,

1. Rules of product
   $1st\ Experiment = p\ total$
   $2nd\ Experiment = q\ total$
   So 1st **and** 2nd experiment : $p\ x\ q\ total$
2. Rules of sum
   $1st\ Experiment = p\ total$
   $2nd\ Experiment = q\ total$
   So 1st **or** 2nd experiment : $p + q\ total$

These basic rules can be expanded for n amount of experiment,

1. Rules of product expanded to
   $p1\ x\ p2\ x\ ..\ x\ pn\ total$
2. Rules of sum expanded to
   $p1 + p2 + .. + pn\ total$

For example, the possibilities to pick 2 persons out of 10 people can be count by using combinatorial as in

$$10\ x\ 9 = 90\ possibilities \qquad (1)$$

The equation (1) can be written as picking 1 out of 10 different people and 1 out of 9 of the same group of people. So there are 90 different combinations to pick 2 different persons out of 10 people.

Apart from this basic rules, there are other theories related to combinatorial,

1. Inclusive - exclusive principal
   This principal is a application of venn diagram which is,

$$|\cup| = || + || - |\cap| \qquad (2)$$

2. Permutation
   Permutation is the total arrangement of certain objects into a sequence. Basic form of permutation is

$$P(n,k) = \frac{n!}{(n-k)!} \qquad (3)$$

3. Combination
   Combination is a specific form of permutation that the arrangement of the sequence doesn't matter. Basic form of combination is

$$C(n,r) = \frac{n!}{r!(n-r)!} \qquad (4)$$

4. Permutation and combination with identical objects
   To arrange a sequence of objects that have 2 or more identical members have different formulas. The 2 identical members can't be swapped to add more arrangement. So formula for permutation with repetition is the following,

$$P(n,r) = \frac{n!}{a1\ x\ a2\ x\ a3\ x\ ..\ x\ ar} \qquad (5)$$

   In equation (4), a1, a2, a3, or ar are a total of identical objects. For combination with identical objects has the following formula,

$$C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!} \qquad (6)$$

For this instance, there are a total of 52 cards in one deck that are used to play Blackjack. By using combinatorics we can count the total combination of receiving 2 cards that have a sum value of 21 is

$$\begin{aligned}
&Total\ card\ that\ have\ value\ of\ 10 && = 4 \\
&Total\ face\ card(Jack, Queen, King) && = 3\ x\ 4\ =\ 12 \\
&Total\ Ace\ cards && = 4 \\
&(4+12)\ x\ 4\ = 64\ combinations && \quad (7)
\end{aligned}$$

Meanwhile, the combinations to receive any 2 different card is

$$C(52,2) = \frac{52!}{2!50!} = 1326 \qquad (8)$$

So there are 64 out of 1326 different combinations of receiving a 10, Jack, Queen, or King and an Ace in the beginning of a round of Blackjack, the value of the cards will be explained in the next section.

Then with this principal, we can count the possibility of any cards that we need to win. The player can use these possibilities to consider the multiple decisions and strategy while playing Blackjack. The strategy to play Blackjack variates, depends on the combination of cards the player gets. To determine the strategy of playing Blackjack, first we have to learn how to play.

III. How To Play / Rules

Blackjack requires two sides, the dealer and the player. The player's goal is just to try to win against the dealer in every round, this goal is different when you are in a casino. Meanwhile the dealers don't have a goal, their purpose is only to deal the cards to the player and themself, then let the player decide their own action.

Cards with different faces have different values, all numbered cards from 2-10 have their respective face value, for example a 3 of club card has a value of 3. The Ace cards have either 1 or 11 value, this value can change depending on the hands. Meanwhile the face cards, Jack, Queen, and King cards all have a value of 10. So to get a sum value of 21, the player needs an Ace and a 10 or any face card. The type of the card doesn't matter in Blackjack, so an 8 of club and an 8 of heart aren't different. For example if the player was dealt a 10 of club and a 7 of spade, the player's hand currently has a "Hard" value of 17. Another example if the player was dealt an Ace of diamond and a 9 of club, that means the player has a "Soft" value of 19 or 10. The term "Soft" is used when you have accumulated some value with an Ace card in your hand, because an Ace card can be used as a 1 or 11. If the player didn't have any Ace card it will use the term "Hard" for the hand's values. Then you have the term "Blackjack" if you have accumulated exactly 21 from your hands, for example an Ace and any face card.

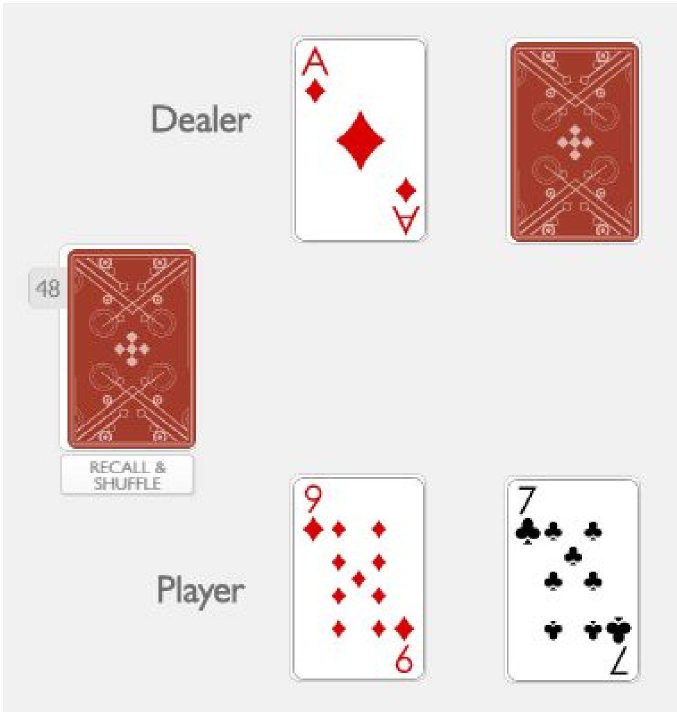

Fig. 3.1, Example of a beginning of a round of Blackjack

This paper will use the basic rules of Blackjack with no variation and no option to "Double", because bets aren't going to be considered a factor. The game starts with the dealer deals 2 cards to the player and to himself/herself. Player's cards could be face-up or face-down depending on the place, in this paper the cards will be dealt face-up. Meanwhile, the dealer will only show 1 of the cards face-up and the other face-down. So the player will only be working on information about the cards in their hand and one of the dealer cards. In the following are the conditions of losing and winning.

If the player is dealt an Ace and a 10 value card, the player is granted “Blackjack” or “natural”, because they gain a sum of 21 with 2 cards only. The player wins if the dealer doesn’t get a “Blackjack”.

Fig. 3.2 Example of player getting “Blackjack” with an Ace of heart and a King of spade

If the player exceeds a sum of 21, the player busts and loses instantly, regardless of the dealer’s cards. But if the dealer exceeds 21 or busts, the player wins if the player doesn’t busts.

If the player accumulates a sum higher than the dealer’s hand without busting, the player wins and the contrary. Then if both dealer and player receive a Blackjack or gain the same sum of cards, no one wins.

After the cards are dealt, the player can choose between these options, Hit or Stand. As mentioned in the previous section, the player can gain more value by choosing to hit. If the sum is too close to busts, then the player can choose to Stand and continue to the dealer’s turn. The dealer isn’t provided with the same options. The dealer must keep hitting until their value is hard 17 or above, or until it busts. If the dealer gets an Ace, they can still continue hitting because the value is considered soft.

An additional option is available when you are dealt 2 same value cards in your hand, it is called “Split”. This paper won’t include this option, because this decision is focused on the betting part. For the purpose of showing how combinatorics can be a factor to win Blackjack, we will add an option to “Surrender” if the player hands are unwinnable.

IV. Decision Making

To show how combinatorics of the cards could lead to winning a game of Blackjack, this paper will analyze multiple games of Blackjack being played. Decisions that the player makes will be purely based on the result of combinatorics, so there will be no risk taken like a real person would make while playing Blackjack. The player will be safe to stand if the value in hand is between 18 - 21 and the dealer’s possibilities to beat it are small. To help with the illustration, the paper will use a tool from a playing card games creator site[5].

1. *First Game Situation*

Fig 4.1, first game situation(a)

The player hands have a hard value of 10, with a 6 and a 4, while the dealer has a 9. Because the limit is 21 and the highest value we can get from the deck is 10, so all 48 remaining cards are considered safe to hit.

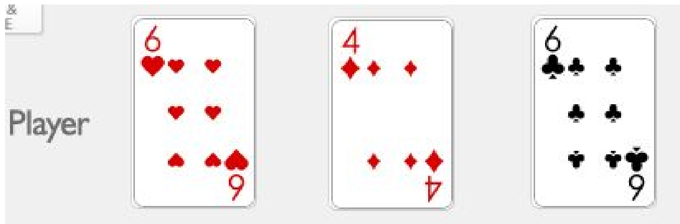

Fig 4.1, first game situation(b)

The player got a 6 that will make the value of hard 16 in the player’s hand. With only 5 more to 21, now we can use some combinatorics. The player is safe with hitting if they get card of 5 or below,

$$Total\ Ace, 2, 3, 5,\ or\ 4\ cards\ =\ (4\ x\ 4)\ +3 =\ 19$$

Out of 47 cards, the player has 19 different cards that can place their hand in a safe place. If the player stands, the only chance of winning is for the dealer to busts. Meanwhile if we check the dealer card of 9, the dealer can get to 17 or higher by having a second card of

$$an\ 8, 9, 10, or\ Ace\ = 4\ x\ 4\ = 16 \text{ out of } 47$$

The remaining 31 possibilities still have the possibility of the dealer busts. If the dealer's second card is a 7, the dealer will bust with a hit of

a 6 to 10 value cards = 2 + 3 + 4 + 3 + 4 = 16 out of 46

The possibilities of the dealer busting keeps going down if the second card is below a 7. So with a maximum of 30 out of 46 cards are safe for the dealer. Because the dealer's chance to win is higher than the player, the safest decision for the player is to surrender the round.

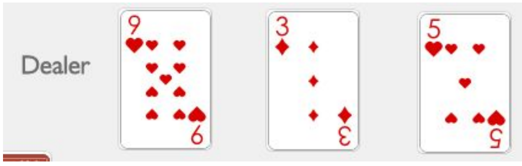

Fig. 4.1, first game situation(c)

In Fig. 4.1(c) the dealer gets a hard 12 then hits to get a final value of 17 beating our 16, so to surrender is the correct move.

2. *Second Game Situation*

Fig. 4.2, second game situation(a)

In the second game, the player has a hard 12 and the dealer has a 7 as shown in Fig. 4.2(a). If the player hits, he/she only busts when the card is 10, so it's 4 out of 48 cards, so the player definitely hit here.

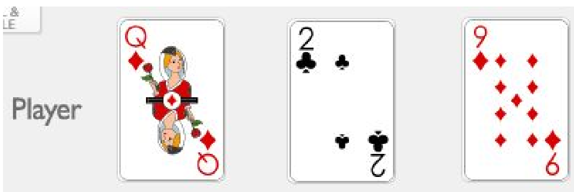

Fig. 4.2, second game situation(b)

With the extra card of 9 in Fig. 4.2(b), the player has accumulated exactly a sum of 21 and achieved Blackjack. There are only 4 possibilities of getting a 9. With a Blackjack, the dealer can only tie it with a Blackjack of their own. To obtain it, the dealer needs 14 with

Combination the sum of 2 cards to get 14 is

- Ace and 3 : 4 x 4 = 16
- 10 or face and 4 : 15 x 4 = 60
- 9 and 5 : 3 x 4 = 12
- 6 and 8 : 4 x 4 = 16
- 7 and 7 : 3 x 2 = 6

The total is 110 out of C(47,2) which is 1081 possibilities. With that probability, it is safe to say that the possibilities for the dealer to beat the player is really small and the player can confidently stand and pass the turn to the dealer.

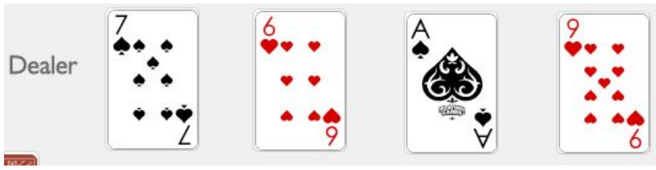

Fig. 4.2, second game situation(c)

In Fig. 4.2(c) the dealer has an original value of 13 then hit to get an Ace making it to 14 because 24 will bust. Because 14 is still under 17, the dealer hit again with a 9 to get a total value of 23. That makes the dealer busts with the value of 23. So the probability checks off and the player's decision is the correct one again.

3. *Third Game Situation*

Fig 4.3, third game situation(a)

In the third game, the player gets a hard 19, meanwhile the dealer gets a hard 6 as shown in Fig 4.3(a). The player is safe to hit if the next card is a 2 or an Ace, so stand is the preferable choice here. If the player stands, no matter the dealer's second card, the dealer will hit because the dealer's hand only has the highest value of hard 16. So we are working with 2 cards combinations here. The player wins if the dealer gets 17,18,19 or busts. The dealer wins if the next two cards sum is 14 or 15.

Combination the sum of 2 cards to get 14 is

- Ace and 3 : 4 x 4 = 16
- 10 or face and 4 : 15 x 4 = 60
- 9 and 5 : 3 x 4 = 12
- 6 and 8 : 3 x 4 = 12
- 7 and 7 : 4 x 3 = 12

The total is 112. With the same method, we count the combination of the sum of 2 cards to get 15 is 101. So the possibilities of the dealer winning is 213 out of C(48,2) which is 1176. So the player only loses on around 20% of the first hit. For the second hit the probability decreases for the dealer to win. So the best move for the player is to stand.

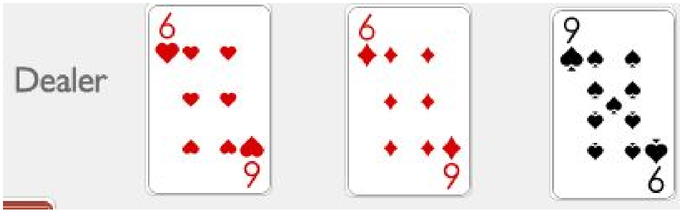

Fig. 4.3, third game situation(b)

Unfortunately, in Fig. 4.3(b) the dealer has a hard 12 then followed with a 9, summing up to 21. Luck still has some factors in this game. The dealer managed to get one of the 213 possibilities out of 1176 combinations. This game situation shows that using combinatorics still has to do with chances. The player still loses even with one of the best hands you can get in the beginning.

4. *Fourth Game Situation*

Fig. 4.4, fourth game situation(a)

For the fourth game as shown in Fig. 4.4a, the player gets a hard 5 facing the dealer's hard 10. With a deficiency of 16, it is safe to say for the player to hit here because all 48 cards aren't going to make the player busts.

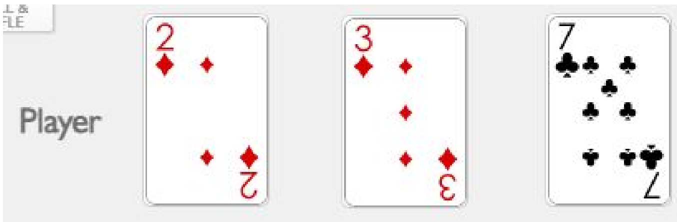

Fig. 4.4, fourth game situation(b)

In Fig. 4.4b, the player got a 7 to make a total value of 12. The player only busts if the next card is a 10 or face card with the possibilities of

a 10 or face cards = 3 + 4 + 4 + 4 = 15 out of 47

With 32 possibilities out of 47, the player is safe and can even make a Blackjack if the next card is a 9. So to hit is the preferred choice here.

Fig. 4.4, fourth game situation(c)

The player manages to get 4 out of 47 possibilities to get a Blackjack to avoid their hand from losing, so standing is a must here. The dealer still has a 10 and can post a Blackjack easily. The dealer can tie this with the second card if the second card is an Ace, with the possibilities of 4 out of 47 the same odd as the player's. But the dealer still automatically lose if the second card is a 7 to 10 value cards with

the possibilities to get a 7-10 value cards :
3 + 4 + 3 + 15 = 26 out of 47 remaining cards

There are more than half of the remaining cards which is a win for the player because those cards will make the dealer get a 17-20 that is still less than the player's 21. So for this instance, the player is in a good spot here.

Fig. 4.4, fourth game situation(d)

The player secures another win when the dealer's second card is an 8 to make the dealer's hand is 18. Because the dealer won't hit anymore if the hand's value is above 17.

IV. Conclusion

In 4 games of Blackjack, the player has won 2 and the dealer has won 2 games but in one of those games, the player correctly surrenders. For the first game, the player was able to surrender because the probability was too favorable for the dealer. Meanwhile in the second and fourth game, the player was granted a good combination of cards from the beginning, so the possibilities were clearly on the player's side. But even when the player dealt good cards, the player still needs to know when to stop and stand. To know when to stand, counting the possibilities of winning and losing can be really helpful. The third game proves that even the smallest possibilities can still happen in this game, so the worst scenario or the best scenario can still happen even when the chance of that happening is below 10 percent.

This paper shows that counting the possibilities of the card by using combinatorial can be a factor while making decisions. People that are able to keep track of the cards in the deck and count them gain more edges than those who don't. But aside from this, Blackjack isn't a fully strategic game. Sometimes risk needs to be taken to win a game of Blackjack even when the possibilities show otherwise.

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PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

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